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Beam optics applications: quantumlike versus classical-like domains

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Abstract. A review of the charged-particle beam optics in terms of recently developed approaches within both classical-like and quantumlike frameworks is presented. On the basis of the mutual connection of optics and mechanics, a brief overview of the quantumlike approach to electron optics is presented. In particular, the main results of the optical applications of the thermal wave model in phase space are given within the Wigner–Weyl picture. Furthermore, the tomographic approach in both classical-like and quantumlike domains in terms of the marginal-probability distribution is also presented. In particular, possible applications of the tomographic approach to optical problems with aberrations for accelerators are put forward. Some aspects of using quantumlike systems in quantum computing projects are discussed.

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1 Introduction

For the last few decades, the importance of describing classical systems, in view of the quantum formalism, have been recognized by researchers in several branches of physics. At the present time, these approaches are known as "quantumlike" descriptions [1]. They have been developed in terms of both linear and nonlinear Schrödinger equations and applied to a number of topics, especially, for describing the coherent effects in optics and the dynamics of charged-particle beams in accelerating machines, electromagnetic traps, nonlinear and collective phenomena in plasma and transmission lines, in condensed matter, etc. The physics involved, which is basically classical, can be recovered by replacing formally the Planck constant with a suitable fundamental parameter of the particular system considered. On the other hand, the proper quantum description, that has been recently applied to several frontiers of physics to describe coherent correlated states, squeezed states, macroscopic coherence in superconductivity, stochastic quantization, mesoscopic gravitation, etc., has been developed in a way entirely similar to the one used for quantumlike models. The profound similarity (in terms of formalism and common methodologies) among the above areas of research has provided for a transfer of know how (in terms of algorithms and many solutions of quantum mechanics) from one branch to another, stimulating the development of each branch in a fairly efficient way. Both macroscopic and mesoscopic coherences described by various quantumlike approaches were proposed recently for charged-particle-beam optics and dynamics, plasma physics, particle and atom trapping, nonlinear optics, tomography techniques, mesoscopic systems (including mesoscopic gravitation), gravitational-wave detection (see [2]).

In this paper, we review and discuss optical methodologies and quantumlike approaches in physics. First of all, we describe the development registered recently in the quantumlike description for optical applications within the mutual connection of optics and mechanics. We also discuss how the quantumlike behavior of classical optical systems, such as light beams in optical fibers and charged beams in electron optics, can be used to simulate some aspects of important projects such as quantum computing.

1.1 The bridge between optics and mechanics

Since its early formulation, the analogy between optics and mechanics has proved to be fruitful in producing important physical insights. For example, it is well-known that this analogy was very important to arrive,

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passing through the construction of wave mechanics, at the present formulation of quantum mechanics that has been recognized as the fundamental theory of nature. It is worth mentioning some important steps of the development of this analogy.

The first analogy put *geometrical optics* in correspondence with *classical mechanics*, on the basis of the similar formulation of the Fermat principle and the Hamilton principle.

The above analogy was quickly recognized to be useful in the study of the charged-particle motion in the presence of electromagnetic fields. The natural development of this branch was the formulation of *electron optics*, which was employed for several scientific and technological applications, such as electron microscopy and particle accelerators. For many years, electron optics remained to be formulated at the level of geometrical optics. Within this framework, the formulation of electron optics can be given in a way very similar to electromagnetic geometrical optics provided one replaces the notion of *light beams* and *refractive index* with *electronic rays* and *potential*, respectively.

The analogy has been extended to the wave level, going from optics to mechanics by de Broglie [3] and Schrödinger [4] and, as a result, wave mechanics was discovered and subsequently quantum mechanics. The *transition* from classical to wave mechanics has been induced by just considering the relationship between geometrical and wave optics. The same kind of transition has been performed by Bohr [5] with a formal procedure called *quantization* based mainly on a set of formal prescriptions called *quantization rules*, where the Planck constant $\hbar$ played a crucial role. These rules allow to go from the classical formulation of mechanics to another formulation in terms of operators; in such a way, to obtain an evolution equation for the physical system under consideration.

Within the framework of the above procedure, when the transition from classical to quantum mechanics has been performed, the Schrödinger equation was recognized as the nonrelativistic limit of a more general wave-mechanical formulation induced by the correspondence with wave optics [6]. In fact, the nonrelativistic limit of the Klein–Gordon equation, which was a certain correspondence with the d'Alambert equation, is just the Schrödinger equation.

Going back from quantum mechanics to wave optics, the above nonrelativistic limit has been considered also for the d'Alambert equation, *i.e.*, for the Helmholtz equation, by Fock and Leontovich [7] while considering the problem of radiation-beam propagation through an arbitrary medium. They showed that the equation, which governs this propagation, is a sort of Schrödinger equation, where $\hbar$ and time were replaced by the inverse of the wave number and the propagation coordinate, respectively. The Schrödinger-like equation by Fock and Leontovich was actually obtained from the electromagnetic-wave equation in the paraxial approximation, where slopes of the light rays were considered as very small with respect to the propagation coordinate. It is possible to see that this approximation is equivalent to the so-called slowly-varying amplitude approximation, widely used in nonlinear optics [8] and plasma physics [9], as well as to the nonrelativistic limit of the electromagnetic-wave equation.

The above correspondence, going back from quantum mechanics to wave optics, has been extended more recently by Gloge and Marcuse [10] by performing the transition from geometrical optics to wave optics in a way quite similar to the one *ala Bohr*. In the formal quantization of Gloge and Marcuse, a set of quantization rules (in which $\hbar$ and time are replaced by the inverse of the wave number and the propagation coordinate, respectively) are introduced in the Hamiltonian for the electromagnetic rays. The result is the electromagnetic-wave equation whose limit, in the paraxial approximation, gives the Fock–Leontovich equation.

1.2 Further developments

The procedure of Gloge and Marcuse turns out to be very fruitful because it provides a way of transferring algorithms and many solutions of quantum mechanics to radiation-beam physics, especially, for optical fibers [11,12], coherent- and squeezed-state theories [13,14], Schrödinger cat states (like even and odd coherent states [15] and superposition states created in Kerr medium [16]), and phase-space investigations within a Wigner-like picture [17] where a quasiclassical distribution, very similar to the quantum Wigner transform [18], governed the paraxial electromagnetic-ray evolution. In recent years, the importance of describing, in a unified way, optics of light and optics of electronic rays has been recognized [19] and the possibility to transit from *geometrical electron optics* to *wave electron optics*, has been pointed out [20] as a development of electron optics.

In recent years as well, a procedure *ala Gloge and Marcuse* has been introduced in electron optics to describe the collective behavior of charged-particle-beam transport [21–24]. By using some correspondence rules, called *thermal quantization rules*, in which $\hbar$ and time are replaced by the *beam emittance* [25] and the propagation coordinate, respectively, a quantumlike description of electronic rays called the thermal wave model (TWM) can be constructed. This procedure, applied in the paraxial approximation, gives rise to a Schrödinger-like equation for a complex function, the so-called beam wave function whose squared modulus is proportional to the beam density.

A novel approach, which consists of a *deformation* of the phase-space equation for electronic rays, allows the TWM to be recovered, in a way alternative to the one *ala Gloge and Marcuse*, but only in the semiclassical approximation [26]. The method has been later applied to paraxial electromagnetic beams [27]. The *transition* from the classical description to the quantumlike description allows to obtain a von Neumann-like equation which, in turn, provides a Wigner-like description of the charged-particle-beam transport in the semiclassical approximation. In the next section, we briefly present TWM and the main results obtained in phase space for the optical application, beyond the semiclassical approximation.

2 Thermal wave model

According to this model, the beam transport is described in terms of a complex function, the *beam wave function* (BWF), whose squared modulus gives the transverse density profile. This function satisfies a Schrödinger-like equation, in which the Planck constant $\hbar$ is replaced by the transverse emittance ϵ. In this equation, the potential term, say U, accounts for the total interaction between the beam and the surroundings. In particular, in a particle accelerator, for the potential, one has to take into account both the multipole-like contributions (depending only on the machine parameters) and the collective terms which, on the contrary, depend on the particle distribution (self-interaction).

In transverse dynamics, TWM has been applied to a number of linear and nonlinear problems. In particular, it seems to be capable of reproducing the main results of the Gaussian particle-beam optics (dynamics for a quadrupole-like device [21]), as well as estimating the luminosity in the final focusing stages of linear colliders in the presence of small aberrations [22,23]. In addition, for the case of transverse dynamics, in quadrupole-like devices with small sextupole and octupole deviations, the TWM predictions have been compared with tracking-code simulations and a fair agreement has been demonstrated [24].

Let us consider a charged-particle beam travelling along the z-axis with velocity βc ($\beta \approx 1$) and having transverse emittance ϵ. If we denote with x the transverse coordinate (1D case), within the TWM framework, the transverse-beam dynamics is ruled by the following Schrödinger-like equation [21]:

$$\mathrm{i}\epsilon \frac{\partial \Psi}{\partial z} = -\frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}\Psi + U(x,z)\Psi. \tag{1}$$

Note that $U(x,z)$ is a dimensionless energy potential, obtained by dividing the potential energy associated with the transverse-particle motion by the factor $m_0\gamma\beta^2c^2$, where m_0 and γ are the particle rest mass and the relativistic factor $[1-\beta^2]^{-1/2}$, respectively. The z-constancy of the integral $\int_{-\infty}^{+\infty} |\Psi(x,z)|^2 \,\mathrm{d}z$, which is a consequence of the reality of $U(x,z)$ in (1), suggests the interpretation of $|\Psi(x,z)|^2$ as the transverse-density profile of the beam. Hence, if N is the total number of the beam particles, then $\lambda(x,z) \equiv N\ |\Psi(x,z)|^2$ is the transverse-number density.

2.1 Phase-space descriptions by means of Wigner function

According to the quantum formalism, for a given beam wave function $\Psi(x,z)$, one can introduce the density matrix ρ as

$$\rho(x,y,z) \equiv \Psi(x,z)\Psi^*(y,z), \tag{2}$$

which, in Dirac's $\langle bra|$ and $|ket\rangle$ notation, is associated with the following *density operator*

$$\hat{\rho} = |\Psi\rangle\langle\Psi|. \tag{3}$$

Note that $\hat{\rho}$ has the following two properties:

$$\text{(i) probability conservation} \qquad \mathrm{T}r(\hat{\rho}) = 1; \tag{4}$$

$$\text{(ii) hermiticity} \qquad \hat{\rho}^\dagger = \hat{\rho}. \tag{5}$$

On the basis of this density-matrix definition, one can define the relevant phase-space distributions associated with the transverse-beam motion within the framework of TWM.

One of the widely used phase-space representations given in quantum mechanics is the one introduced by Weyl and Wigner. In this representation, simply by replacing $\hbar$ with ϵ, the phase-space particle-beam dynamics can be described in terms of the following function, called the Wigner function:

$$\rho_w(x,p,z) \equiv \frac{1}{2\pi\epsilon}\int_{-\infty}^{+\infty} \rho\left(x-\frac{y}{2},\, x+\frac{y}{2},\, z\right) \times \exp\left(\mathrm{i}\,\frac{py}{\epsilon}\right)\mathrm{d}y, \tag{6}$$

namely, by virtue of (2), for a pure state,

$$\rho_w(x,p,z) = \frac{1}{2\pi\epsilon}\int_{-\infty}^{+\infty} \Psi^*\left(x+\frac{y}{2},\, z\right)\Psi\left(x-\frac{y}{2}, z\right) \times \exp\left(\mathrm{i}\,\frac{py}{\epsilon}\right)\mathrm{d}y. \tag{7}$$

From (7), it is easy to prove that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho_w(x,p,z)\mathrm{d}x\mathrm{d}p = 1. \tag{8}$$

Let us introduce the quantities

$$\lambda(x,z) = N\int_{-\infty}^{+\infty} \rho_w(x,p,z)\mathrm{d}p \tag{9}$$

and

$$\eta(p,z) = N\int_{-\infty}^{+\infty} \rho_w(x,p,z)\mathrm{d}x. \tag{10}$$

Relations (9) and (10) show that $N\rho_w(x,p,z)$ is the phase-space-distribution function associated with the transverse-beam motion. In fact, ρ_w satisfies the following von Neumann-like equation [24]:

$$\frac{\partial \rho_w}{\partial z} + p\,\frac{\partial \rho_w}{\partial x} + \frac{\mathrm{i}}{\epsilon}\left[U\left(x+\frac{\mathrm{i}\epsilon}{2}\frac{\partial}{\partial p}\right) - U\left(x-\frac{\mathrm{i}\epsilon}{2}\frac{\partial}{\partial p}\right)\right]\rho_w = 0. \tag{11}$$

Consequently, $\lambda(x,z)$ and $\eta(p,z)$ are configuration-space and momentum-space projections of $N\rho_w(x,p,z)$, respectively. Although $\rho_w(x,p,z)$ is the distribution function of the system within the framework of TWM, due to well-known quantum-mechanical properties, it is not always positive. However, in quantum mechanics, it is positive for some special wave functions of the harmonic oscillator called *coherent states* [13,28,29], which give purely Gaussian density profiles.

Coherent states for charged-particle beams have been recently introduced in TWM in order to describe the coherent structures of charged-particle distributions produced in an accelerating machine [30]. The fact that ρ_w can assume negative values for some particular cases reflects the quantumlike nature of both the wave function and the density operator. In fact, we can easily find the following quantumlike uncertainty relation [21,26]:

$$\langle x^2\rangle\langle p^2\rangle \geq \frac{\epsilon^2}{4} = \text{const.}, \quad (12)$$

where $\langle x^2\rangle \equiv \sigma^2(z)$ and $\langle p^2\rangle \equiv \sigma_p^2(z)$ are the r.m.s of the configuration- and momentum-space distributions, respectively.

2.2 Optical problems with aberrations

A numerical study has been pursued [24] in order to compare the description of the phase-space as given by TWM, both by means of ρ_w and by means of the Husimi function (*i.e.*, the Q-function [31]), with the one resulting from standard particle tracking.

A flat Gaussian (1D) particle beam has been used as the starting beam, with emittance $\epsilon = 120\times10^{-6}$ m rad, $\sigma_0 \equiv \sigma(0) = 0.05$ m, and $\sigma_{p0} \equiv \sigma_p(0) = 1.2\times10^{-3}$ rad.

A simple device made of a quadrupole magnet plus a drift space has been considered as a beam-transport line; in addition, sextupole and/or octupole aberrations have been included in the quadrupole. It was supposed that, at $z = 0$, the beam enters a focusing quadrupole-like lens of length l with small sextupole and octupole deviations, then it propagates *in vacuo*. In this region, the beam particles feel the following potential:

$$U(x,z) = \begin{cases} \frac{1}{2!}k_1x^2 + \frac{1}{3!}k_2x^3 + \frac{1}{4!}k_3x^4 & 0 \leq z \leq l \\ 0 & z > l \end{cases}, \quad (13)$$

where k_1 is the quadrupole strength, k_2, the sextupole strength, and k_3, the octupole strength, respectively.

The Wigner function (7) and the Q-function have been computed by numerical integration for different combinations of aberration strengths and have been compared with the results of tracking of 7×10^5 particles [24]. Isodensity contours at σ, 2σ, and 3σ have been used to describe the particle distribution in the phase space, both before and after the passage through the simple device specified above. It is worth noting that, with this choice, only 2% of the particles were found beyond the contour at 2σ, and only 0.01% of them are beyond the contour at 3σ. A fair agreement with tracking was observed for the contours at σ and 2σ for both ρ_w and Q-function. The contours at 3σ showed some discrepancies, more pronounced in the case of the Wigner function, which in the periphery of the distribution produces regions with negative phase-space density, that yielded an unrealistic distortion of the phase space [24]. By using the Q-function, instead, this effect was largely smoothed out [24]. These large values of distortion approach the limits of applicability of perturbation theory which, for the starting parameters and the quadrupole strength selected, are given by $D \ll 3{\sigma_0}^2K_1/\epsilon = 2.25$ (for the sextupole perturbation) and $D \ll 4{\sigma_0}^2K_1/\epsilon = 3.00$ (for the octupole perturbation), where D is the phase-space distortion as defined in reference [24].

In spite of the discrepancies observed at large amplitudes due to the particularly strong perturbations used in this study, these results can be considered as fairly satisfactory — the TWM phase-space description of beam dynamics in the presence of a nonlinear lens and, in particular, the one given by the Q-function is in more than reasonable agreement with the one given by classical accelerator physics for all realistic values of perturbation.

3 Tomographic techniques for particle beams

3.1 Marginal distribution for charged-particle beams in classical domain. A Fokker–Planck-like equation

Within the framework of the classical-like description of charged-particle beams, we can introduce a map of the beam wave function onto the positive probability distribution using formulas adopted from the noncommutative tomography approach [32] and from tomographic probabilities in quantum mechanics [33].

The tomographic map was also used for a purely classical Boltzmann equation [34,35]. Since in accelerator physics the classical Fokker–Planck equation plays an important role [36], below we discuss the tomographic map for the classical distribution function obeying the Fokker–Planck equation. This means that we introduce the symplectic tomography representation [37,38] for the Fokker–Planck equation.

Let us discuss a general problem of classical statistics for a distribution, which evolves according to the Fokker–Planck equation. We consider the equation for the marginal distribution function $w\,(X,\,\mu,\,\nu,\,t)$ for the case where the classical probability distribution function $\rho\,(q,\,p,\,t)$ satisfies the Fokker–Planck equation.

In view of the notation $p = x_1$ and $q = x_2$, the Fokker–Planck equation reads [39]

$$\frac{\partial\rho}{\partial t} = \sum_{i,\,j=1}^{2}\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}\left(x_1,x_2\right)\rho - \sum_{i=1}^{2}\frac{\partial}{\partial x_i}D_i\left(x_1,x_2\right)\rho, \quad (14)$$

where D_{ij} and D_i are the diffusion matrix and the linear drift vector, respectively. The marginal distribution $w\,(X,\,\mu,\,\nu,\,t)$ relates to the probability distribution $\rho\,(q,\,p,\,t)$ by means of the symplectic tomography transform

$$w\,(X,\,\mu,\,\nu,\,t) = \int \exp\left[-\mathrm{i}k(X-\mu q-\nu p)\right] \times \rho\,(q,\,p,\,t)\;\frac{\mathrm{d}k\,\mathrm{d}q\,\mathrm{d}p}{(2\pi)^2}, \tag{15}$$

$$\rho\,(q,\,p,\,t) = \frac{1}{2\pi}\int w\,(X,\,\mu,\,\nu,\,t) \times \exp\left[-\mathrm{i}\,(\mu q+\nu p-X)\right]\,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{d}X. \tag{16}$$

Relations (15) and (16) provide the following correspondence rule for the action of some operators onto the distribution $\rho\,(q,\,p,\,t)$ and the marginal distribution $w\,(X,\,\mu,\,\nu,\,t)$:

$$q\,\rho\,(q,\,p,\,t) \longrightarrow -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}\,w\,(X,\,\mu,\,\nu,\,t)\,,$$

$$\frac{\partial}{\partial q}\,\rho\,(q,\,p,\,t) \longrightarrow \mu\,\frac{\partial}{\partial X}\,w\,(X,\,\mu,\,\nu,\,t)\,,$$

$$p\,\rho\,(q,\,p,\,t) \longrightarrow -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\,w\,(X,\,\mu,\,\nu,\,t)\,,$$

$$\frac{\partial}{\partial p}\,\rho\,(q,\,p,\,t) \longrightarrow \nu\,\frac{\partial}{\partial X}\,w\,(X,\,\mu,\,\nu,\,t)\,. \tag{17}$$

The above correspondence rules mean that the Fokker–Planck equation rewritten for the marginal distribution $w\,(X,\,\mu,\,\nu,\,t)$ is given by

$$\frac{\partial w}{\partial t} = \left[\left(\frac{\partial}{\partial p}\right)^2 D_{11}\,(p,q) + \left(\frac{\partial}{\partial q}\right)^2 D_{22}\,(p,q) +2\frac{\partial}{\partial p}\frac{\partial}{\partial p}D_{12}\,(p,q)\right]w - \left[\frac{\partial}{\partial p}D_1\,(p,q) + \frac{\partial}{\partial q}D_2\,(p,q)\right]w, \tag{18}$$

where we make the replacement

$$\frac{\partial}{\partial p} = \nu\,\frac{\partial}{\partial X}, \qquad \frac{\partial}{\partial q} = \mu\,\frac{\partial}{\partial X},$$
$$p = -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu} \qquad q = -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}\cdot$$

We consider a partial case of equation (18) for the constant symmetric diffusion matrix $D_{ij}\,(p,q) = D_{ij},\ \ (i,\,j=1,\,2)$ and the linear drift vector $D_i\,(p,q) = -\sum_{j=1}^{2}\gamma_{ij}\,x_j$. For this case, the Fokker–Planck equation describes the Ornstein–Ulenbeck process and it can be written in a compact form corresponding to the equation for a quadratic system [40]. To do this clearly, we introduce four-vector operators Q_α and M_α $(\alpha = 1,\,2,\,3,\,4)$

$$\mathbf{Q} = \begin{pmatrix}\partial/\partial p\\ \partial/\partial q\\ p\\ q\end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix}\nu\,(\partial/\partial X)\\ \mu\,(\partial/\partial X)\\ -\,(\partial/\partial X)^{-1}\,(\partial/\partial\nu)\\ -\,(\partial/\partial X)^{-1}\,(\partial/\partial\mu)\end{pmatrix}. \tag{19}$$

Equation (14) then takes the form

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\mathbf{Q}B\mathbf{Q}\rho + c\rho; \qquad \mathbf{Q}B\mathbf{Q} = \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4} Q_\alpha B_{\alpha\beta} Q_\beta. \tag{20}$$

The constant 4×4-matrix B is expressed in terms of the constant matrix D_{ij} and the constant matrix γ_{ij} $(i,\,j=1,\,2)$

$$B = 2\begin{pmatrix} D_{11} & D_{12} & \gamma_{11}/2 & \gamma_{12}/2\\ D_{21} & D_{22} & \gamma_{21}/2 & \gamma_{22}/2\\ \gamma_{11}/2 & \gamma_{21}/2 & 0 & 0\\ \gamma_{12}/2 & \gamma_{22}/2 & 0 & 0\end{pmatrix} \quad \text{and} \quad c = \frac{\gamma_{12}+\gamma_{22}}{2}\cdot \tag{21}$$

We arrive at equation (18) in the form

$$\frac{\partial w}{\partial t} = \frac{1}{2}\,\mathbf{M}B\mathbf{M}w + cw. \tag{22}$$

For particle beams in accelerators, the model of the Fokker–Planck equation in the classical approximation can be described by the same equation (14) for the positive density with replacement of the function $\rho\,(q,\,p,\,t)$ by the function $\rho\,(q,\,p,\,\epsilon,\,z)$, where ϵ is emittance; with the time derivative being replaced by the z-derivative. On the other hand, for the particle beam, equation (22) is valid for the positive marginal distribution $w\,(X,\,\mu,\,\nu,\,\epsilon,\,z)$, in which the emittance and z-dependences are added. One can point out that in the quantumlike picture we have an equation for the Wigner quasidistribution. If this equation is taken in the form of the Fokker–Planck equation (14), the corresponding marginal distribution satisfies the same evolution equation (18). This is, in fact, done in Section 3.3.

3.2 Particle-beam tomography in the quantumlike domain

In Section 2, we have shown that the quantumlike picture naturally provides for a full phase-space description in terms of a Wigner-like distribution. But, unfortunately, as in quantum mechanics, this distribution can be negative and it does not match with the usual classical picture that can be given in particle-beam physics in terms of a positive-finite probability distribution. We address the following question: how to make a bridge between the two descriptions? First of all, note that also in the quantumlike approach there is a possibility to switch from the classical phase-space equation to an equation for a positive marginal distribution, which has standard classical features. In fact, in quantum optics and quantum mechanics, both the optical tomography method [41,42] and the symplectic tomography method [37,38] were suggested for measuring quantum states.

3.3 The Fokker–Planck-like equation for the quantumlike marginal distribution

In this section, we review a recently proposed approach to obtain the evolution equation describing the particle beam in terms of a Fokker–Planck-type equation for the positive probability distribution function. We also obtain this beam-evolution equation for an arbitrary potential [35]. To these ends, let us first of all note that:

(1) it is well-known from quantum mechanics that the Wigner function [18] represents the non-negative density operator $\widehat{\rho}$ [43] in a particular representation. It is Hermitian, *i.e.*, $\widehat{\rho}^\dagger = \widehat{\rho}$ and $\mathrm{Tr}\,\widehat{\rho} = 1$;
(2) for any representation, diagonal elements of the density operator are non-negative, since they describe the probability distribution function in a corresponding basis;
(3) in the coordinate representation, we have $\langle x|\widehat{\rho}|x\rangle = P(x)$, with $P(x)$ being the position distribution function. The Wigner-like function ρ_w, which satisfies equation (11), is related to the density matrix in the coordinate representation by an invertible transform (hereafter, we take $\epsilon = 1$) [24]:

$$\rho_w\,(q,\,p) = \int \left\langle q + \frac{u}{2} \mid \widehat{\rho} \mid q - \frac{u}{2} \right\rangle \exp\left(-\mathrm{i}pu\right) \mathrm{d}u, \tag{23}$$

$$\langle x \mid \widehat{\rho} \mid x' \rangle = \frac{1}{2\,\pi} \int \rho_w \left(\frac{x + x'}{2},\, p \right) \mathrm{e}^{\mathrm{i}p(x-x')}\, \mathrm{d}p; \tag{24}$$

(4) on the basis of reference [44], it is possible to prove that, for any Hermitian operator $\widehat{X}$, the Fourier transform of a characteristic function $\chi(k) \equiv \langle \exp(\mathrm{i}k\widehat{X})\rangle = \mathrm{Tr}\widehat{\rho}\exp(\mathrm{i}k\widehat{X})$, *i.e.*, $w(y) = \frac{1}{2\pi}\int \chi(k)\exp(-\mathrm{i}ky)\mathrm{d}k$ is the distribution function with classical features.

In fact, by taking into account the positivity of the diagonal elements of the density operator, one can see that $w(y) = \langle y \mid \widehat{\rho} \mid y\rangle \geq 0$ and $\int w(y)\,\mathrm{d}y = 1$. By considering a specific operator $\widehat{X} = \mu\hat{q} + \nu\hat{p}$, one can write that

$$w\,(X,\,\mu,\,\nu) = \int \rho_w(q,\,p)\ \mathrm{e}^{-\mathrm{i}k(X-\mu q-\nu p)}\ \frac{\mathrm{d}k\,\mathrm{d}q\,\mathrm{d}p}{(2\pi)^2}, \tag{25}$$

this means that $w\,(X,\,\mu,\,\nu) \geq 0$ and $\int w\,(X, \mu, \nu)\,\mathrm{d}X = 1$.

By analogy with quantum optics, we call this function the quantumlike marginal distribution of the particle beam. Note that we have used here the property of the Wigner distribution function

$$\mathrm{Tr}\,\widehat{\rho}\,\mathrm{e}^{\mathrm{i}k(\mu\hat{q}+\nu\hat{p})} = \int \rho_w\,(q,\,p)\,\mathrm{e}^{\mathrm{i}k(\mu q+\nu p)}\ \frac{\mathrm{d}q\,\mathrm{d}p}{2\,\pi}\cdot \tag{26}$$

Formula (25) can be inverted

$$\rho_w(q,\,p) = \frac{1}{2\pi}\int w\,(X,\,\mu,\,\nu)\,\mathrm{e}^{-\mathrm{i}(\mu q+\nu p-X)}\ \mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{d}X. \tag{27}$$

For beam dynamics, one can construct an equation in terms of the marginal distribution following [38]. For the quantumlike Hamiltonian

$$H = \frac{\hat{p}^2}{2} + U(x,z), \qquad \hat{p} = -\mathrm{i}\epsilon\,\frac{\partial}{\partial x}, \tag{28}$$

we obtain from (11), in view of (25), the following Fokker–Planck-like equation for the marginal distribution $w\,(X,\ \mu,\ \nu,\ z,\ \epsilon)$ of the form (for a beam of arbitrary emittance ϵ) [35,45]:

$$\frac{\partial w}{\partial z} - \mu\frac{\partial}{\partial \nu}w + \frac{\mathrm{i}}{\epsilon}\left[U\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu} + \mathrm{i}\frac{\nu\epsilon}{2}\frac{\partial}{\partial X}\right)\right.$$
$$\left. - U\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu} - \mathrm{i}\frac{\nu\epsilon}{2}\frac{\partial}{\partial X}\right)\right] w = 0. \tag{29}$$

One can write equation (29) in the form

$$\widehat{L}w = \frac{2}{\epsilon}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\,U^{(2n+1)}\,(\hat{q})\left(\frac{\nu\epsilon}{2}\right)^{2n+1}\frac{\partial^{2n+1}w}{\partial X^{2n+1}}, \tag{30}$$

where

$$\widehat{L} \equiv \frac{\partial}{\partial z} - \mu\frac{\partial}{\partial\nu} - \nu U^{(1)}(\hat{q})\frac{\partial}{\partial X}, \tag{31}$$

with

$$U^{(2n+1)}(\hat{q}) \equiv \frac{\partial^{2n+1}}{\partial x^{2n+1}}U(x=\hat{q}) \quad \text{and} \quad \hat{q} = -\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu}\cdot$$

Note that equation (30) is completely analogous to equation (11). In particular, for a quadrupole (harmonic potential well), $U = k_1x^2/2$ (k_1 being the quadrupole strength), equations (30) and (11) become, respectively,

$$\left\{\frac{\partial}{\partial z} - \mu\frac{\partial}{\partial\nu} + k_1\nu\frac{\partial}{\partial\mu}\right\} w = 0$$
$$\text{and}\quad \left\{\frac{\partial}{\partial z} + p\frac{\partial}{\partial x} - k_1 x\frac{\partial}{\partial p}\right\}\rho_w = 0. \tag{32}$$

However, the important difference is that equation (30) is written for the positive-probability-distribution function within the framework of the standard description of classical stochastic processes used in the usual probability theory. The Gaussian solutions to equation (32) have the following form:

$$w\,(X,\mu,\nu,z) = \frac{1}{\sqrt{2\pi\sigma_X(z)}}\ \exp\left\{-\frac{(X-\bar{X})^2}{2\sigma_X(z)}\right\},$$
$$\bar{X} = \mu\langle q\rangle + \nu\langle p\rangle. \tag{33}$$

In view of the above discussion, it is not a great deal to draw a conclusion that a quantumlike behavior of a charged-particle beam in an accelerator may be described by the positive-probability-distribution function. The extra parameters μ and ν describe an ensemble of reference frames in the phase space.

The above results answer the question formulated earlier. Moreover, note that the information coded by the beam wave function of TWM can be presented in the form of the distribution function (marginal distribution). In fact, the suggested description in terms of the marginal distribution $w(X, \mu, \nu, z)$ provides an expression for the positive probability distribution *via* the Wigner transform; moreover, the quantumlike context, in which this is done, keeps all the quantumlike effects, including the quantumlike uncertainty relation (12).

Within the TWM framework, we find the beam wave function, then calculate the Wigner transform and, *via* a Fourier transform, the marginal distribution. Alternatively, we could emphasize that the main property of marginal distribution consists of using a rotation in phase space as a tool for a complete beam-state measurement. In principle, we can avoid speaking in terms of the Wigner function [18] (or in terms of other quasidistributions, widely used in quantum mechanics and quantum optics [13,28,29]). We can start from the classical single-particle physics, where the potential is given, and go directly to the Fokker–Planck–like equation (29) for the positive-probability-distribution function, which incorporates all the quantumlike effects of TWM. From the computation point of view, solving this Fokker–Planck-like equation for practical interesting cases (for example, sextupoles, octupoles, etc.) in physics is similar to solving the Schrödinger equation or von Neumann–Moyal equation in quantum mechanics, or to solving the conventional Fokker–Planck equation in classical physics.

While solving directly equation (29), one avoids numerical errors introduced by intermediate transforms on the way from the BWF to the marginal distribution.

4 Entanglement and quantumlike computation possibilities

As we demonstrated the quantumlike behavior of optical and particle beams can simulate the quantum behavior of microsystems in many aspects. In view of this, an important possibility emerges that the quantum elements and processes used in quantum computing projects can be also simulated by classical (but quantumlike) systems [46]. Below we discuss shortly some features of this possibility. To do this, we again clarify what it means for a classical system to be a "quantumlike."

Quantumlike systems are classical systems in which classical processes are described by the mathematical equations which are identical to the mathematical equations describing quantum processes in quantum systems. The most important point about quantum computation by quantumlike systems is that all features of the mathematical algorithm of quantum computation may be implemented by quantumlike systems, without falling into contradiction with the non-equivalence of classical and quantum mechanics, because unitary evolution in time is replaced by unitary evolution in one space coordinate. Light rays propagating in optical fibers can be considered as one example of quantumlike systems. Sound propagating along the specific water layers in ocean and detected by a submarine sonar is another example of a quantumlike system. Over the last half century, paraxial description of electromagnetic waves in nonlinear media (optical fibers, plasmas, etc.), paraxial description of charged-partical-beam transport, signal-analysis formulation, image reconstruction and related tomographic representations have been successfully formulated by means of the quantumlike formalism and applied to a number of physical problems. Typically, a quantumlike system is governed by a suitable Schrödinger-like equation and/or a density matrix equation in which Planck's constant is replaced with some other characteristic parameter connected to the physical nature of the system itself. We have discussed this property in the previous sections. An important property of quantum systems is entanglement of two subsystems.

How does entanglement work in quantumlike system?

Entanglement in quantum systems corresponds to a superposition of wave functions of different particles or different degrees of freedom. The entanglement means that different quantum particles can influence each other much more strongly than classical particles. Entanglement in a classical system corresponds to a superposition of mode functions depending on different degrees of freedom, *e.g.*, the x- and y-modes of electromagnetic radiation propagating along z-direction in an optical fiber. The mathematical description of both quantum and classical entanglements is identical.

Thus, the question arises — How would a quantum computer be physically implemented in a quantumlike system? How would it be different from a quantum computer based on quantum particles?

A quantum computer in quantumlike systems can be physically constructed as a combination of the following elements. Let us take an example of concrete light beams propagating in optical fibers. There are pieces of optical fibers with a specially prepared profile of refractive index. The technology of such preparation today is well developed. The pieces are connected into an optical line which is an analog of hard ware in today's standard computer. Light emitted by a laser (lasers) propagates through the line and is detected at the output of the line. Numerical information coded at the input of the line by the phase and intensity profile of electromagnetic field is elaborated (process of computation) by propagating through the line (chain) of the optical fibers with appropriately prepared refractive index profiles. At the line output, the light beam again in a coded mode contains the result of the computations (*e.g.*, Fourier decomposition of some signals, solution of an equation, etc.). To read out information, one can use a detector measuring the classical light beam state. This measurement is completely analogous to measuring the wave function or density matrix in quantum mechanics and it can be provided by using, for example, the tomography of electromagnetic signal. After measuring, decoding gives the result of computation.

A quantum algorithm implemented on a quantumlike system is identical (mathematically, not physically!) to the same algorithm implemented on a fully quantum computer. A computer has to provide purely mathematical operations (no matter whether or not it is a fully quantum computer, fully classical or classical quantumlike one). In a quantum computer, one uses the notion of *qubit* (*i.e.* quantum bit) of information. In the quantumlike case, the notion of *qulbits* (*i.e.* quantumlike bit) emerges [46], which is a complete analog of qubit in the quantum case.

In the case of a physical realization of a quantum algorithm on a quantumlike system, such aspects as parallelism and efficiency will be of the same level as in a quantum computer.

The quantumlike "quantum" computer can be used as a simple and understandable model of a quantum computer for investigating the properties of its elements. If one is fortunate, it can be even an alternative to computing by means of real quantumlike systems.

The technical implementation in fiber optics of the gates, not necessarily a full computer, seems in reach of today's technology.

The challenges to be overcome are the final understanding of the basic principles of quantum computations and finding the best technological solutions for materials providing the appropriate profile of refractive index for the quantumlike computer's elements.

The charged-particle beams also have properties to be used for quantumlike computation.

5 Conclusions and remarks

In this paper, we have reviewed and discussed the main results of the beam optics in terms of the recently developed classical-like and quantumlike approaches. We have used the formal analogy of the equations of quantum mechanics and quantum optics to the equations of the thermal wave model describing electronic rays in charged-particle-beam optics. Additionally, in view of this analogy, the new results obtained in quantum optics were translated into the electron-optics domain. Thus, we have constructed the tomographic representation for the classical Fokker–Planck equation [Eq. (18)], including the particular case of the Ornstein–Ulenbeck process.

We conclude that, in charged-particle-beam transport, the quantumlike description within the framework of the thermal wave model is a useful tool and the tomography approach can be applied successfully in the same manner as in quantum optics. In fact, it seems that the new tomographic-probability description of charged-particle beam optics can provide additional tools to the study of charged-particle beam transport in several devices, such as linear and circular accelerating machines and electromagnetic traps. Quantumlike models are successfully employed in signal–analysis problems (see, *i.e.*, [47]). Also the quantumlike description of optical beams (both light beams in fibers or electron beams in accelerators or plasma) can be used to simulate several aspects of the quantum computing process including entanglement and quantum control of information processing.

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